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STATISTICAL ANALYSIS OF BRITTLE SOLID DAMAGE BASED ON NETWORK MODEL

JIANQIANG FAN and GUANGQUAN ZHOU
Department of Modern Mechanics, University of Science and Technology of China, Hefei Anhui 230026, People's Republic of China

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Abstract—A theoretical model of microcrack nucleation and growth is proposed based on the network model. The corresponding damage evolution law of the brittle solid under constant tensile stress is obtained by means of mesoscopic statistical analysis. Furthermore, a statistical strength distribution of damaged materials is deduced according to the weakest link concept. The results indicate that the strength distribution is related to the damage extent in the materials and nonlinearly dependent on the local stress concentration at the crack tip. A theoretical relation between the damage and the average strength of the damaged material is proposed.

1. INTRODUCTION

The failure of a brittle solid under tensile stress is due to a large amount of microcrack nucleation, growth and coalescence in the body of the material (Curran *et al.*, 1987). Although extensive interest exits in this area, the investigations are insufficient not only on the mesoscopic dynamic laws concerning nucleation, growth and coalescence but also on the macroeffects of damage.

To deal with the great number of the microcracks appearing in the damaged material, statistical analysis methods were proposed and regarded as effective ways to study this complicated phenomenon (Curran *et al.*, 1987; Bai *et al.*, 1988; Xing, 1986). Comparably, the mesoscopic statistical analysis method proposed by Bai *et al.* (1988) expresses more physical essence of the brittle solid damage. At the early stage of damage, according to the conservation of microcrack number in a phase space, Bai *et al.* (1988, 1992) derived the governing equation of the statistical evolution of a one-dimensional system with ideal microcracks, which approximately satisfies the following partial differential equation

$$\frac{\partial n}{\partial t}+\frac{\partial(n\dot{c})}{\partial c}=\dot{n}_N \tag{1}$$

where $n(c, t)$ is the number density of microcracks, c is the length scale variable, $\dot{c}$ the growth rate and $\dot{n}_N$ the nucleation rate. In using equation (1), one should determine the laws of the nucleation rate and the growth rate. However, up to now, the definition of nucleation and growth has not been proposed clearly (Curran *et al.*, 1987). The nucleation laws and the growth laws available now are based more on the fitting of experimental results than the theoretical consideration.

In the last few years, the network model has been proposed in theoretical physics and has been widely used to investigate the phenomena of fracture and breakdown of disordered materials (Herrmann and Roux, 1990; Meakin, 1991). However, most work concentrated on the numerical simulation such as the investigation of fractal properties of cracks (Hinrichsen *et al.*, 1989) and the critical behaviors of fracture (Schwartz *et al.*, 1985). The theoretical analysis is rare even in a simple network model (Meakin, 1991).

Fortunately, efforts have been made to model the damage and breakdown in a network system theoretically (Curtin and Scher, 1991, 1992, 1993). In the initial work by Curtin and Scher (1991, 1992), it is found that the microcrack distribution in a damaged network tends to an algebraic form when η, a nonlinear exponent, gets larger. The method in the study may be called a discrete method. However, it does uncover some significant features underlying the damage at mesoscopic level, as discussed by the authors.

A similar attempt is made in the present paper, but using an alternative approach based more on mechanical consideration. The attempt is to seek an effective theoretical method to model the damage at mesoscopic level and to derive the effects or responses of damage at macroscopic level. The network model is chosen for its simplicity and intuitiveness. A similar approach has been used by Ostoja-Starzewski *et al.* (1994) in their system work on damage of random materials. The model introduced here may be called a continuum method. Some similar results are obtained to those by Curtin and Scher (1992). Moreover, an explicit microcrack nucleation and growth rate law is derived based on network model. This may be used as theoretical reference in damage mechanics, which is currently lacking. With the mesoscopic statistical analysis method, an analytical expression of microcrack distribution is derived. This made it possible to study the strength distribution of damaged material according to the weakest link concept. Based on these results, a theoretical fitting relation between the damage and the average strength of the damaged material is proposed, which has some engineering value and is currently rare.

2. NUCLEATION AND GROWTH MODEL

Consider a model material described by a network with potential microcracks, as illustrated in Fig. 1. Each solid line segment indicates a potential microcrack. The area enclosed by the potential microcracks stands for a mesoscopic material unit, e.g. a grain. The strength of the grain boundary can be expressed by a spring that connects the two centers of neighboring grains. The spring is indicated by a dash line segment in Fig. 1. So the nucleation of a potential microcrack can be represented by the break of the corresponding spring. Therefore, the network formed by potential microcracks is a mapping of the network made up of springs. Firstly, we assume that the nucleation and the growth are activation processes. The activation of an isolated potential microcrack is called nucleation. The activation of those potential microcracks neighboring an existing crack is

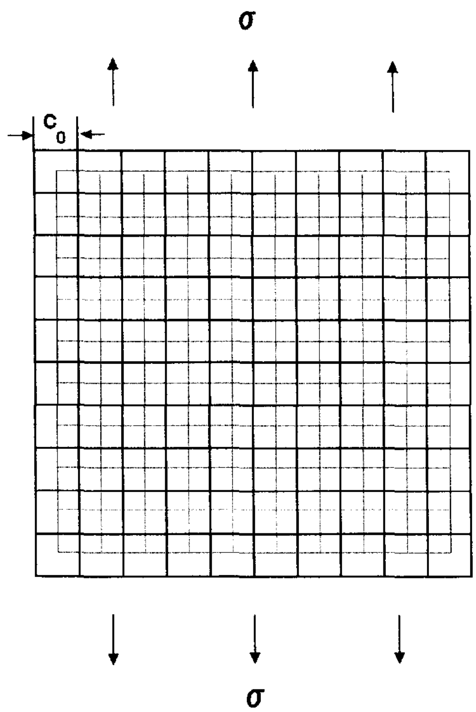

Fig. 1. Model material described by a network. Each solid line segment indicates a potential microcrack. Each dashed line segment stands for a spring.

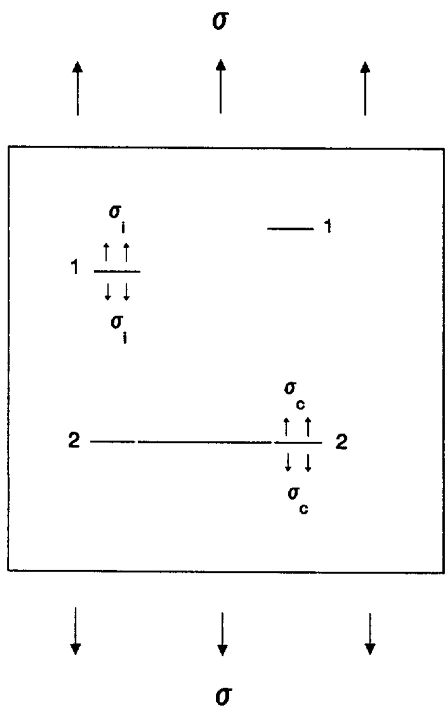

Fig. 2. Nucleation and growth of microcracks. The activation of the potential microcracks labeled 1 is called nucleation; the activation of those labeled 2 is called growth or extension.

called the growth or the extension of the crack as shown in Fig. 2. Secondly, we assume that the activation rate of each potential microcrack is proportional to the stress acting on the microcrack raised to a power η, an exponent of nonlinear extent, which is inspired by the DLA (Diffusion Limited Aggregation) model and has been used in numerical simulation and analysis by Hansen *et al.* (1990) and Curtin and Scher (1991, 1992).

With the above assumptions and definitions, the activation rate of an isolated potential microcrack, f^i, is

$$f^i = A\left(\frac{\sigma_i}{\sigma_0}\right)^\eta \tag{2}$$

where η is nonlinear exponent, σ_i is the tensile stress acting on the ith isolated potential microcrack as shown in Fig. 2, σ_0 is a material parameter and A is a geometric factor. Because only one type of microcrack exists in the network system, A is a constant factor. Thus the nucleation rate of this system is

$$\dot{n}_N = A\sum_{i=1}^{N(t)}\left(\frac{\sigma_i}{\sigma_0}\right)^\eta \delta(c-c_0), \tag{3}$$

where $N(t)$ is the number of all isolated potential microcracks per unit volume at time t, c_0 is the critical length of the nucleated microcrack and δ is the Dirac delta function. At the early stage of damage, the number of the nucleated microcracks is small. The nucleated microcracks distribute sparsely in the body of the material. So not only the interaction of the microcracks can be ignored but also the stress acting on each isolated potential microcrack, σ_i, can approximately be considered as equal to the applied tensile stress σ. $N(t)$ can approximately be replaced by N_T, the number of potential microcracks per unit volume at initial time. So eqn (3) can be expressed as

$$\dot{n}_N = AN_T\left(\frac{\sigma}{\sigma_0}\right)^n \delta(c-c_0). \tag{4}$$

As soon as a potential microcrack is nucleated, it has the ability of extension. Similarly, the growth rate of an existing crack with length c can be written as

$$\dot{c} = Zc_0A\left(\frac{\sigma_c}{\sigma_0}\right)^n \tag{5}$$

where Z is the number of potential microcracks at the tip of the crack which can be activated, and σ_c is the enhanced stress acting on the potential microcracks at the crack tip, as shown in Fig. 2. In the early stage of damage, according to Griffith's theory, the enhanced stress σ_c and the applied tensile stress σ have the following relation:

$$\sigma_c = a\sigma\sqrt{c}, \tag{6}$$

where a is also a constant geometric factor, for the length over which the enhanced stress is averaged remains the same, which is c_0 in the model. Therefore, the growth rate is

$$\dot{c} = Zc_0Aa^n\left(\frac{\sigma}{\sigma_0}\right)^n c^{n/2} \quad c > c_0. \tag{7}$$

This growth rate law is accordant with the growth rate obtained by Evans and Wiederhorn (1984) based on their experimental results for ceramics.

3. THE CHARACTERISTIC OF DAMAGE EVOLUTION

With the nucleation law eqn (4), and the growth law eqn (7), we can give the governing equations of the damage evolution at the early stage under a constant tensile stress, i.e.

$$\frac{\partial n}{\partial t} + \frac{\partial(n\dot{c})}{\partial c} = \dot{n}_N$$

$$\dot{n}_N = AN_T\left(\frac{\sigma}{\sigma_0}\right)^n \delta(c-c_0)$$

$$\dot{c} = Zc_0Aa^n\left(\frac{\sigma}{\sigma_0}\right)^n c^{n/2} \quad c > c_0. \tag{8}$$

If we nondimensionalize the variables in the above equations with the characteristic variables c_0, A, N_T and σ_0, we can get the following dimensionless variables:

$$\tilde{c} = \frac{c}{c_0}, \quad \tilde{t} = tA, \quad \tilde{n} = \frac{n}{N_T}, \quad \tilde{a} = a\sqrt{c_0}$$

$$\tilde{\sigma} = \frac{\sigma}{\sigma_0}, \quad \tilde{\dot{c}} = \frac{\dot{c}}{c_0A}, \quad \tilde{\dot{n}}_N = \frac{\dot{n}_N}{AN_T}. \tag{9}$$

The dimensionless governing equations are

$$\frac{\partial \tilde{n}}{\partial \tilde{t}} + \frac{\partial(\tilde{n}\tilde{\dot{c}})}{\partial \tilde{c}} = \tilde{\dot{n}}_N$$

$$\tilde{\dot{n}}_N = \tilde{\sigma}^n \delta(\tilde{c}-1)$$

$$\dot{\tilde{c}} = Z\tilde{a}^{\eta}\tilde{\sigma}^{\eta}\tilde{c}^{\eta/2} \quad \tilde{c} \geqslant 1. \tag{10}$$

Because of the discontinuity at $\tilde{c} = 1$, we first seek the solution of $\tilde{n}(1, \tilde{t})$, which satisfies the following equation:

$$\frac{\mathrm{d}\tilde{n}(1,\tilde{t})}{\mathrm{d}\tilde{t}} = \tilde{\sigma}^{\eta} - \frac{\eta}{2}Z\tilde{a}^{\eta}\tilde{\sigma}^{\eta}\tilde{n}(1,\tilde{t}). \tag{11}$$

The first term on the right side of eqn (11) is the inflow term and the second is the outflow one. So we can get an analytical expression of $\tilde{n}(1, \tilde{t})$,

$$\tilde{n}(1,\tilde{t}) = \frac{2}{Z\eta\tilde{a}^{\eta}}\left[1-\exp\left(-\frac{\eta}{2}Z\tilde{a}^{\eta}\tilde{\sigma}^{\eta}\tilde{t}\right)\right]. \tag{12}$$

When $\tilde{c} \geqslant 1$, we suppose the distribution of microcrack is continuous, while $\tilde{n}_N = 0$. So eqn (12) can be regarded as the boundary condition of eqn (10), i.e.

$$\begin{aligned}
&\frac{\partial\tilde{n}}{\partial\tilde{t}} + \frac{\partial(\tilde{n}\dot{\tilde{c}})}{\partial\tilde{c}} = \tilde{n}_N \\
&\dot{\tilde{c}} = Z\tilde{a}^{\eta}\tilde{\sigma}^{\eta}\tilde{c}^{\eta/2} \quad \tilde{c} \geqslant 1 \\
&\tilde{n}(1,\tilde{t}) = \frac{2}{Z\eta\tilde{a}^{\eta}}\left[1-\exp\left(-\frac{\eta}{2}Z\tilde{a}^{\eta}\tilde{\sigma}^{\eta}\tilde{t}\right)\right].
\end{aligned} \tag{13}$$

According to the partial differential equation theory, the solution of the above equations is

$$\begin{aligned}
\tilde{n}(\tilde{c},\tilde{t}) &= \frac{p}{q}\frac{1}{\tilde{c}^{m}}\{1-\exp[-q(\tilde{t}-t_0)]\} \\
t_0 &= \frac{1}{q}\ln(\tilde{c}) \quad m = 1 \\
t_0 &= \frac{m}{q(1-m)}(\tilde{c}^{1-m}-1) \quad m \neq 1,
\end{aligned} \tag{14}$$

where $m = \eta/2$, $p = \tilde{\sigma}^{\eta}$, $q = (\eta/2)Z\tilde{a}^{\eta}\tilde{\sigma}^{\eta}$ and $\tilde{t} \geqslant t_0$. In this paper, we suppose $Z = 2$, $\tilde{a} = 1.3$ and $\tilde{\sigma} = 1.5$. Because

$$t_0 = \int_1^{\tilde{c}} \frac{\mathrm{d}\tilde{c}}{\dot{\tilde{c}}},$$

the inequality $\tilde{t} > t_0$ means that a minimum time t_0 must be needed in order to observe the number density of a crack with length c. This implies that a crack with maximum length, $c_{\max}$, exists at any given time, which satisfies the following relation:

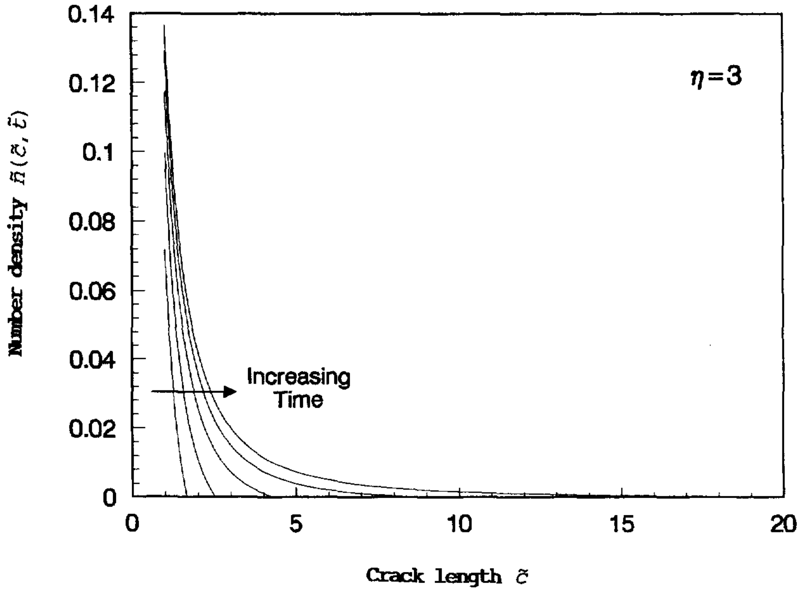

Fig. 3. Distribution of number density of microcracks at different time ($\tilde{t} = 3\times10^{-2}$, 5×10^{-2}, 7×10^{-2}, 9×10^{-2}, 11×10^{-2}).

$$\tilde{t} = \int_1^{\tilde{c}_{max}} \frac{d\tilde{c}}{\dot{\tilde{c}}}. \tag{15}$$

A typical number density distribution of microcracks is shown in Fig. 3, from which we can see that at any time there are more small cracks than large ones. This result is qualitatively consistent with the practical situation.

The damage function $\tilde{D}(\tilde{t})$ is defined as

$$\tilde{D}(\tilde{t}) = \int_0^\infty \tilde{c}\tilde{n}(\tilde{c}, \tilde{t})\, d\tilde{c} \tag{16}$$

which can be understood as the total dimensionless crack area per unit volume. The definition is similar to the one in continuum damage mechanics. With the microcrack distribution, eqn (14), we can get the characteristic of damage evolution, as shown in Fig. 4, where we can see damage rate varies with the exponent η. The larger the η is, the higher damage rate the system has.

In general, η can be understood as "brittleness degree" of a material. The more brittle a material is, the larger η it has. However, under dynamic loading, because of the rate effect, many kinds of mesoscopic heterogeneity can be activated. As a result, a single crack has not enough time to grow and this leads to a decrease of η in the same system. Therefore, η is related to both the material property and the loading condition.

In order to understand the rules governing the microcrack distribution, we first observe the microcrack distribution in different η system with the same damage extent. From Fig. 5 we can see with the increase of η the distribution is closer to an algebraic distribution form. Next, we observe the microcrack distribution with various damage as shown in Fig. 6, from which we can see that the heavier the damage is, the closer to the algebraic form the distribution is.

4. MACROEFFECTS OF THE DAMAGE

One macroeffect produced by damage is the scattering of the strength of the damaged material. The statistical distribution law can be obtained by using the weakest link principle

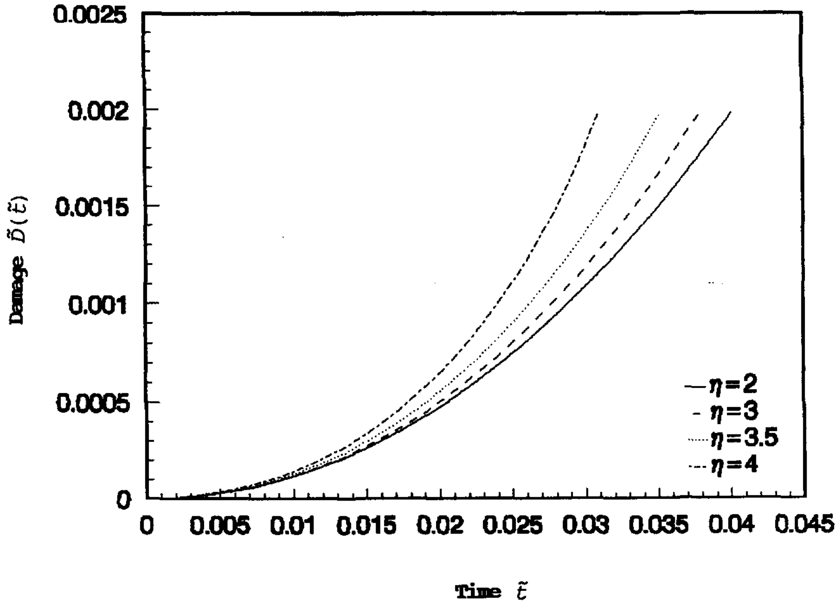

Fig. 4. The characteristic of damage evolution. The larger the η is, the larger damage rate the corresponding system has.

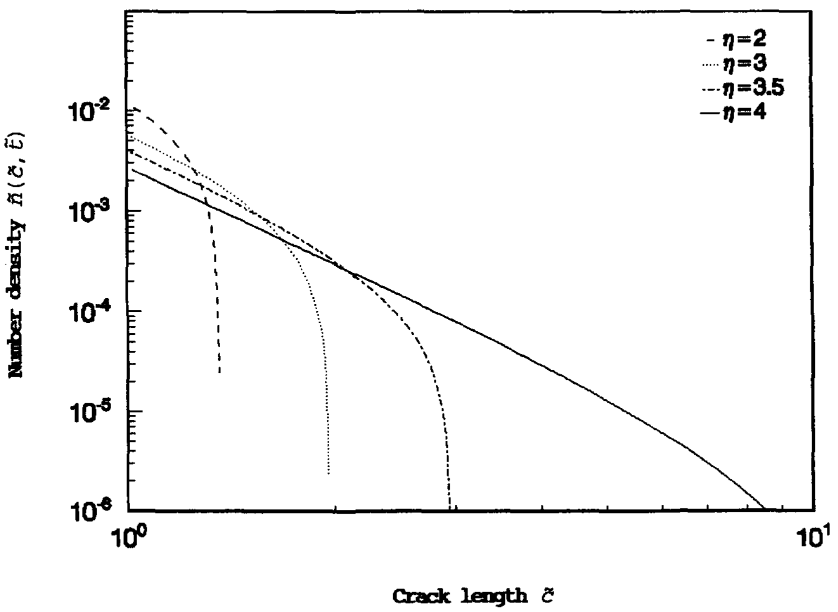

Fig. 5. Distribution of number density of microcracks, in different η system with the same damage $\tilde{D} = 2\times10^{-3}$. The larger the η is, the closer to the algebraic form the distribution is.

and the concept of probability density of microcracks (Fan and Zhou, 1994). Generally, the strength distribution under tensile stress has the following form:

$$P_f(\sigma_s) = 1-\exp(-VF(\sigma_s)) \tag{17}$$

where $P_f(\sigma_s)$ is the failure probability under applied stress σ_s, V is the volume of the material, and $F(\sigma_s)$ is the failure probability per unit volume material under stress σ_s. The form of $F(\sigma_s)$ is governed by the distribution of the microcracks in the local material, i.e.

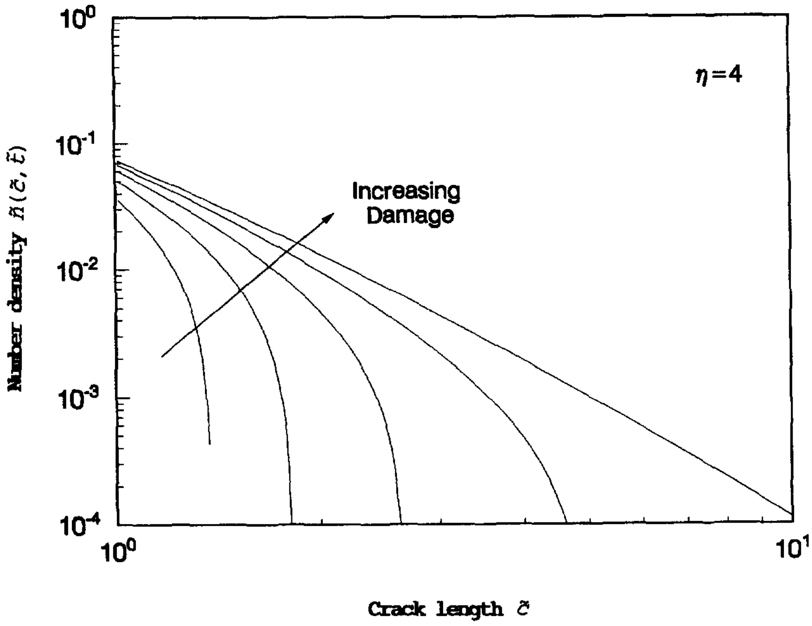

Fig. 6. Distribution of number density of microcracks with different damage extent. The heavier the damage is, the closer to the algebraic form the distribution is ($\tilde{D} = 1.3 \times 10^{-4}, 3.6 \times 10^{-4}, 7.6 \times 10^{-4}, 1.4 \times 10^{-3}, 2 \times 10^{-3}$).

$$F(\sigma_s) = \int_0^{\sigma_s} \rho(\sigma_s)\,d\sigma_s = \int_{c_1}^{\infty} \rho(c)\,dc, \tag{18}$$

where $\rho(\sigma_s)$ is the failure probability density, $\rho(c)$ is the probability density of microcracks, and c_1 is the critical length of the crack that can extend under stress σ_s. If all cracks are assumed to be penny shaped and oriented perpendicularly to the applied stress, then the relation between the crack strength σ_s and the critical length of the crack c_1 is simply

$$\sigma_s = \frac{K_{IC}}{\pi\sqrt{c_1}} \tag{19}$$

where K_{IC} is the toughness of the material.

Defining the dimensionless variables as

$$\tilde{\sigma}_s = \frac{\sigma_s \pi \sqrt{c_0}}{K_{IC}}, \quad \tilde{c}_1 = \frac{c_1}{c_0}, \quad \tilde{\rho} = \rho c_0 \quad \tilde{F} = F c_0^3, \quad \tilde{V} = \frac{V}{c_0^3} \tag{20}$$

we can get the dimensionless form of eqns (17)–(19), i.e.

$$P_f(\tilde{\sigma}_s) = 1 - \exp(-\tilde{V}\tilde{F}(\tilde{\sigma}_s)) \tag{21}$$

$$\tilde{F}(\tilde{\sigma}_s) = \int_{\tilde{c}_1}^{\infty} \tilde{\rho}(\tilde{c})\,d\tilde{c} \tag{22}$$

$$\tilde{\sigma}_s = \frac{1}{\sqrt{\tilde{c}_1}}. \tag{23}$$

The number density of microcracks and the probability density of microcracks have the following relation:

$$\tilde{\rho}(\tilde{c},\tilde{t}) = \frac{\tilde{n}(\tilde{c},\tilde{t})}{\int_0^{\infty} \tilde{n}(\tilde{c},\tilde{t})\,\mathrm{d}\tilde{c}}. \tag{24}$$

The physical meaning of eqn (24) is when a material subjected to an applied stress $\tilde{\sigma}$ undergoes a time $\tilde{t}$, the probability density of the microcrack in the damaged material is $\tilde{\rho}(\tilde{c},\tilde{t})$. The purpose of this section is to derive the strength distribution of the damaged material with a microcrack distribution of $\tilde{\rho}(\tilde{c},\tilde{t})$. According to equation (14), $\tilde{\rho}(\tilde{c},\tilde{t})$ can be expressed as

$$\tilde{\rho}(\tilde{c},\tilde{t}) = \frac{q}{p}\left[\frac{m}{q\tilde{t}+\exp(-q\tilde{t})-1}\right]\tilde{n}(\tilde{c},\tilde{t}). \tag{25}$$

Substituting equations (22), (23) and (25) into eqn (21), we can get the strength distribution law

$$P_{\mathrm{f}}(\tilde{\sigma}_{\mathrm{s}},\tilde{t}) = 1-\exp\left\{-\frac{\tilde{V}}{q\tilde{t}+\exp(-q\tilde{t})-1}[Q+\exp(-Q)-1]\right\}$$

$$Q = q\left[\tilde{t}-\frac{1}{q}\ln(\tilde{\sigma}_{\mathrm{s}}^{-2})\right] \quad m=1$$

$$Q = q\left[\tilde{t}-\frac{m}{q(1-m)}(\tilde{\sigma}_{\mathrm{s}}^{2(m-1)}-1)\right] \quad m\neq 1 \tag{26}$$

where $Q \geqslant 0$, which results from the inequality $\tilde{t} \geqslant t_0$ in equation (14). When $m > 1$,

$$\tilde{t} \leqslant \frac{m}{q(m-1)},$$

which ensures an infinitive crack can not appear in the damaged material under the applied stress $\tilde{\sigma}$ undergoing a time of $\tilde{t}$. The meaning of function (26) is that when a material subjected to an applied tensile stress $\tilde{\sigma}$ undergoes a time of $\tilde{t}$, the remaining strength of the damaged material obeys the distribution function of $P_{\mathrm{f}}(\tilde{\sigma}_{\mathrm{s}},\tilde{t})$. In this paper, we suppose $\tilde{V} = 10$.

When $\tilde{\sigma}_{\mathrm{s}} = 0$, $P_{\mathrm{f}}(\tilde{\sigma}_{\mathrm{s}},\tilde{t}) = 0$, which indicates the material cannot be fractured; when $\tilde{\sigma}_{\mathrm{s}} \to \infty$, $P_{\mathrm{f}}(\tilde{\sigma}_{\mathrm{s}},\tilde{t}) = 1$, which indicates the breakdown will take place in the system. Because the damage varies monotonously with the time, as described by eqn (16), the time can be regarded as a kind of measurement of the damage. Therefore, eqn (26) tells us that the strength distribution of the damaged material varies with the damage. In other words, the strength distribution of the damaged material depends on the damage extent in the material. In the case of lower damage extent, from eqn (26) we can see Q is small. The strength distribution is mainly controlled by $\exp(-Q)$. This indicates the strength obeys a two exponential distribution function, a kind of Duxbury–Leath distribution form (Duxbury *et al.*, 1987). In the case of heavy damage, Q is large while $\exp(-Q)$ is small. The strength distribution is mainly governed by Q, which shows that the strength obeys a kind of Weibull distribution function (Weibull, 1951). These two cases can be seen easily from Fig. 7. Figure 7 also tells us that the average strength decreases with the increase of the damage. On the other hand, in different η systems with the same damage extent, the higher the η, the larger the Q. Therefore the strength distribution is closer to the Weibull distribution function. The lower the η, the smaller the Q. So the strength distribution tends to a Duxbury–Leath distribution form. These two situations can be seen from Fig. 8. The strength distribution laws are consistent with the microcrack distribution laws as stated in Section 3, for the implicit microcrack distribution underlying the Weibull strength distribution is a kind

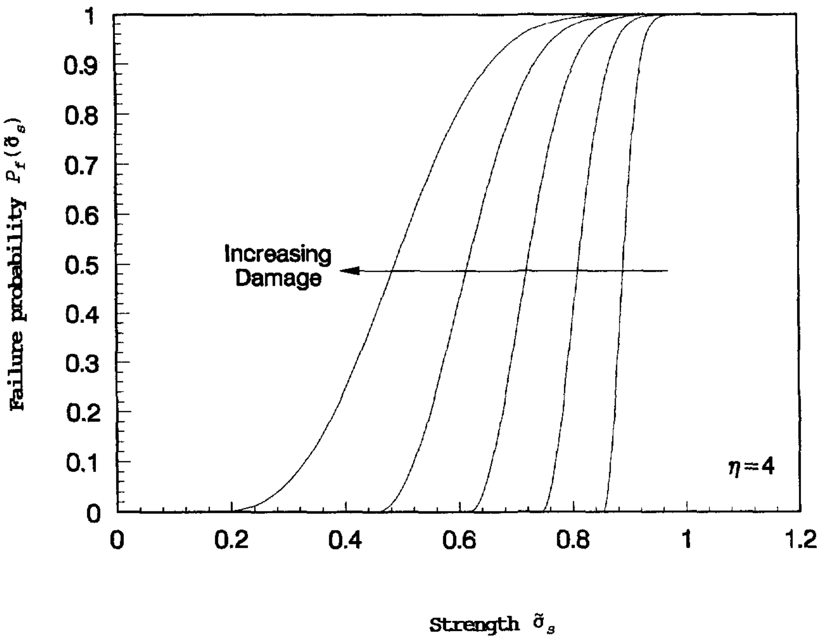

Fig. 7. Strength distribution of damaged material with different damage extent. In the case of heavy damage, the distribution is close to the Weibull distribution form; in the case of light damage, the distribution tends to the Duxbury–Leath distribution form ($\tilde{D} = 1.3\times10^{-4}, 3.6\times10^{-4}, 7.6\times10^{-4}, 1.4\times10^{-3}, 2\times10^{-3}$).

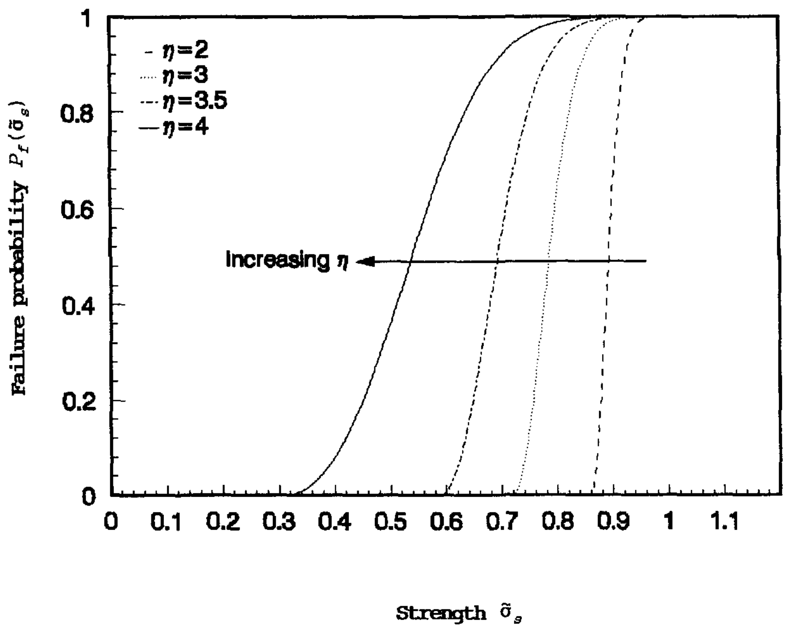

Fig. 8. Strength distribution of damaged material, in different η system with the same damage $\tilde{D} = 2\times10^{-3}$. The larger the η is, the closer to the Weibull form the distribution is.

of algebraic distribution form, while the implicit microcrack distribution underlying the Duxbury–Leath distribution is a kind of exponential distribution forms (Fan and Zhou, 1994).

Another macroeffect of the damage is the decadence of the average strength of the damaged material and one purpose of studying the damage at the mesoscopic level is to investigate the strength of the damaged material. The quantitative relation available between the damage and the strength of damaged material is rare now. As stated above, in a damage system, the time can be regarded as a measurement of damage. So we can obtain

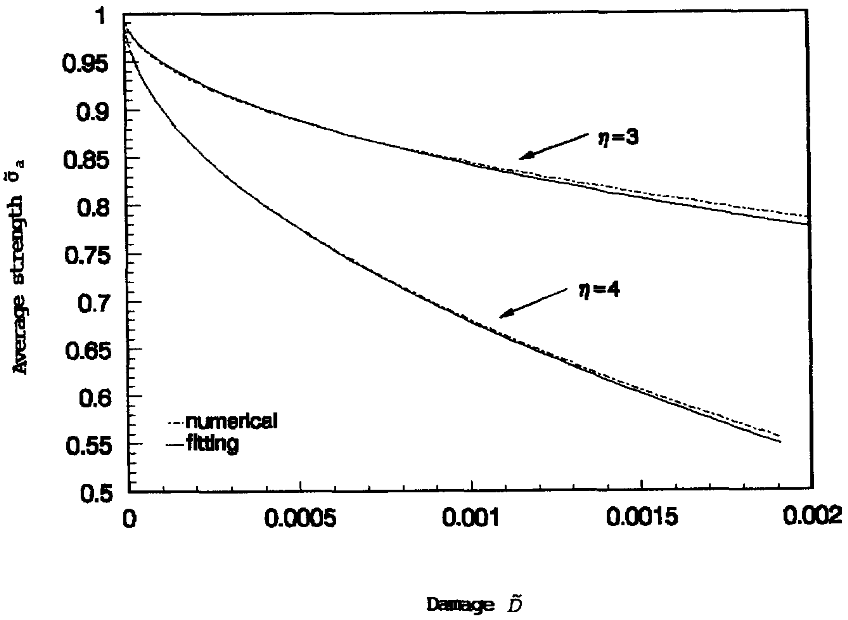

Fig. 9. Schematic relation between the damage and the average strength of the damaged material, which can be fitted with the formula $\tilde{\sigma}_a = 1-\mu\tilde{D}^{\lambda}$.

a parameter equation concerning the damage and the average strength of the damaged material with the parameter $\tilde{t}$, i.e.

$$\tilde{\sigma}_a(\tilde{t}) = \int_0^1 \tilde{\sigma}_s \, dp_f = \int_0^{\infty} (1-P_f(\tilde{\sigma}_s, \tilde{t})) \, d\tilde{\sigma}_s$$

$$\tilde{D}(\tilde{t}) = \int_0^{\infty} \tilde{c}\tilde{n}(\tilde{c}, \tilde{t}) \, d\tilde{c}, \tag{27}$$

where $\tilde{\sigma}_a$ is the average dimensionless strength of the damaged material. This parameter equation generally cannot produce an explicit analytical expression, but we can give a schematic relation of numerical result as shown in Fig. 9. In the convenience of practical usage, we find it can be well fitted with the following formula:

$$\tilde{\sigma}_a = 1-\mu\tilde{D}^{\lambda} \tag{28}$$

where $\lambda = \lambda(\eta)$, $\mu = \mu(\eta)$ are parameters. This formula is open to be confirmed.

5. DISCUSSION AND SUMMARY

The model concerning the microcrack nucleation rate and growth rate introduced in this paper may have both theoretical and practical significance in investigating the damage properties in a network system, e.g. in a spring network (Hansen *et al.*, 1989) or a fuse network (Hansen *et al.*, 1990). The network can be used as a model material of real brittle solid and has been widely studied by the computer simulation in the field of material science and physics. The main disadvantage of the computer simulation is the limitation of the network size. This disadvantage can be overcome by the theoretical analysis method proposed in the paper. Because the network formed by potential microcracks is a mapping of the network formed by the springs, the two systems, when subjected to the same applied stress, have the same damage evolution characteristic. The disadvantage also exits for the theoretical analysis, which is the ignorance of the interaction of the microcracks. Due to the complexity of the microcrack interaction, an effective model of coalescence is not available now. This means the theoretical method can only describe the early stage of damage evolution. However, the simulation method in the network model can give us a perspective of the damage evolution at the later stage. The main advantage of the network simulation method is its natural consideration of the interaction of cracks.

Therefore, the theoretical method and the simulation method are complementary to each other. The former has no size limitation but ignores the crack interaction. The latter has size limitation but considers the interaction of the cracks. The two methods should describe the same damage evolution feature in a damage system at the early stage.

With the intensive work on interaction of microcracks (Kachanov, 1993), it is hopeful that the crack interaction may be considered in the present model in the future, so that the model can describe a general profile of damage at the later stage.

The model presented in this paper holds for $\eta \geqslant 2$, which is applicable to most mechanical systems. In the case of $\eta < 2$, the crack extension rate is weakened according to eqn (7), so the nucleation and coalescence predominate the whole damage process. In the limit case, $\eta = 0$, which is associated with percolation network, there is no crack growth. The damage mechanisms are nucleation and coalescence and the model here loses its power in this case.

Summarizing the theoretical analysis introduced in the paper, we have:

(1) A theoretical nucleation rate and growth rate law has been proposed,

$$\dot{n}_N = A N_{\mathrm{T}} \left(\frac{\sigma}{\sigma_0} \right)^{\eta} \delta(c - c_0)$$

$$\dot{c} = Z c_0 A a^{\eta} \left(\frac{\sigma}{\sigma_0} \right)^{\eta} c^{\eta/2} \quad c \geqslant c_0$$

which can be used as reference for experimental studies of the damage in a brittle material. The damage evolution law in a network system is obtained by the method of mesoscopic statistical analysis.

(2) The damage evolution is nonlinearly dependent on the local stress concentration at the crack tip, which is indicated by the nonlinear exponent η. The large the η is, the closer to the algebraic form the microcrack distribution is. As a result, the strength distribution tends to the Weibull distribution function. As for smaller η, the strength distribution is closer to the Duxbury–Leath distribution function.

On the other hand, the microcrack distribution and the strength distribution vary with the damage. The heavier the damage is, the closer to the algebraic function the crack distribution is. Correspondingly the strength distribution is closer to the Weibull distribution form.

(3) A theoretical fitting relation concerning the damage and the average strength of the damaged material is proposed, i.e.

$$\tilde{\sigma}_{\mathrm{a}} = 1 - \mu \tilde{D}^{\lambda}$$

which needs further verification by experimental studies.

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